At high frequencies we have

$$\alpha_R|_{\omega\to\infty} \sim \frac{3}{2\sqrt{2}} c_f^2 \alpha_T^{(0)} \omega^{-7/2} \langle \xi_j^{-5} \rangle, \quad \alpha_I|_{\omega\to\infty} \sim \frac{3}{2} c_f^2 \alpha_\mu^{(0)} \omega^{-2} \langle \xi_j^{-4} \rangle + \alpha_R$$

Comparison with formulas (6.1) shows that at high frequencies the viscosity of the liquid starts playing a dominant role, suppressing the high-frequency non-linear instability.

In conclusion we note that the LGE theory has been developed much less than the NSE theory, because in general the LGE is a non-integrable equation [3]. Nevertheless, the LGE, like the NSE, arises in the description of many physical systems [3] and is an important object for research, analogies, etc.

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RENORMALIZATION GROUP METHOD FOR THE PROBLEM OF CONVECTIVE DIFFUSION WITH IRREVERSIBLE SORPTION†

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The renormalization group method is used to analyse the propagation of a thin solute slug in a seepage flow with account of diffusion and sorption processes. Sorption is assumed to be partially irreversible and is described by an isotherm with a hysteresis loop. A general technique is developed for analysing the problem. Calculations for the self-similar case are presented and the results are shown to be sufficiently accurate compared with the exact solution.

A NUMBER of problems in the theory of solute transport by seepage flow require consideration of the irreversibility of sorption in the porous medium. Irreversible retention of the solute is particularly

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significant for thin slugs. Irreversibility may have a useful effect for the case of pollutant propagation in groundwater or a harmful effect if the solute is used as a tracer for analysing the seepage flow structure and especially as a vehicle for enhancing oil recovery. In [1, 2], the conventional model of convective diffusion transport of a solute in porous media (see, e.g. [3, 4]) has been applied to allow for irreversible adsorption in a framework of a very simple scheme. Specifically, linear adsorption was assumed with a constant Henry coefficient $da/dt = \Gamma^+$ for $da/dt > 0$ and linear desorption with a different Henry coefficient $da/dt = \Gamma^-$ for $da/dt < 0$; non-equilibrium time-dependent effects were ignored. This adorption "hysteresis" provides a simple description of the experimental data on irreversibility of adsorption at least for a one-time increase/decrease of concentration, which is typical for the tranversal of a thin solute slug. Exact self-similar solutions of the corresponding non-linear model have been constructed in [1, 2] under certain simplifying assumptions: a special law of variation of seepage velocity over time and neglecting the dependence of the diffusion coefficient on the seepage velocity.

In this paper, we examine the same problems using the renormalization group (RG) approach. The main objective of the study is to assess the applicability of the RG technique and to compare the RG results with previous numerical results. This comparison shows that the RG method ensures satisfactory accuracy, so that in future it can be applied to more natural physical situations (including non-self-similar problems).

1. Consider the one-dimensional problem of convective diffusion transport of a thin solute slug by a fluid flow in a porous medium with a constant diffusion coefficient D and different values of the Henry constant for sorption and desorption, i.e. for the regions where the solute concentration respectively increases or decreases. This difference of the Henry constants is introduced to account for the irreversibility of sorption. The solute concentration c in this case evolves according to the equation

$$\frac{\partial (mc + a(c))}{\partial t} + v(t)\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$
$$a'(c) = \begin{cases} \Gamma^+, & \partial c/\partial t > 0 \\ \Gamma^-, & \partial c/\partial t < 0 \end{cases} \tag{1.1}$$

Here $v(t)$ is velocity and $a(c)$ is the quantity of sorbed impurity per unit volume of the medium.

We consider the solution of the Cauchy problem on a straight line for Eq. (1.1), which is continuous together with the solute flux $j = vc - D\partial c/\partial t$. Equation (1.1) can be expressed in an equivalent form [$H(x)$ is the Heaviside function] as

$$\frac{\partial c}{\partial t} + f(t)\frac{\partial c}{\partial x} - \frac{1}{2}\frac{\partial^2 c}{\partial x^2} = -\varepsilon H\left(-\frac{\partial c}{\partial t}\right)\left(f(t)\frac{\partial c}{\partial x} - \frac{1}{2}\frac{\partial^2 c}{\partial x^2}\right) \tag{1.2}$$
$$\varepsilon = (\varepsilon^+ - \varepsilon^-)/\varepsilon^-, \ \varepsilon^\pm = m + \Gamma^\pm, \ f(t) = v(t)/(2D\varepsilon^+)^{1/2}$$

Note that Eq. (1.2) does not have Galilean invariance, because the Heaviside function depends on the Eulerian derivative $(\partial/\partial t)$ and not on the Lagrangian one $(\partial/\partial t + f(t)\partial/\partial x)$. The reason for this is that sorption is a local process and it is determined by the history of variation of the concentration near a given point of the solid porous skeleton at rest, and not in a given moving particle of the liquid. Equation (1.2) is a generalization of the equation of a non-linearly elastic drive introduced in [4] (see also [5]), which has been investigated in detail in a somewhat different statement in [6]. It differs from the previous case by the presence of a convective term.

With the aim of analysing the evolution of a thin concentration impulse, we choose an almost delta-like distribution as the initial condition of the Cauchy problem:

$$c(x, 0) = \frac{Q_0}{\sqrt{2\pi\delta}} \exp\left(-\frac{1}{2\delta}\left(x - \int_{-\delta}^{0} f(s)\,ds\right)^2\right) \equiv Q_0 G(x, 0, -\delta) \quad (\delta > 0) \tag{1.3}$$

where $G(x, t, t_0)$ is Green's function for the convective diffusion equation

$$\left[\frac{\partial}{\partial t}+f(t)\frac{\partial}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right]G(x,t,t_0)=\delta(x)\,\delta(t-t_0) \tag{1.4}$$

Green's function depends on t and t_0 separately, and not on the time lag $(t-t_0)$, because the coefficient in Eq. (1.4) depends explicitly on time. Note that $Q_0G(x,t,t_0)$ is the exact solution of the Cauchy problem for the convective diffusion equation [Eq. (2.2) with $\varepsilon = 0$] if at the initial instant $t=-\delta$ the concentration has a delta-function distribution. Let us investigate the behaviour of the solution in the asymptotic region $t/\delta \rightarrow \infty$.

2. In the general case ($\varepsilon \neq 0$), the solution can be represented in the form

$$c(x,t)=\frac{Q_0}{\sqrt{t}}F\left\{\frac{1}{\sqrt{t}}\left(x-\int_{-\delta}^{t}f(s)\,ds\right),\frac{t}{\delta},\varepsilon\right\} \tag{2.1}$$

It has been shown [1, 2] that, like the problem of a modified heat source [4], problem (1.1) for $v=\lambda t^{1/2}$ and the initial conditions (1.3) does not have a self-similar asymptotic solution of the first kind as $t\rightarrow\infty$ ($\delta\Rightarrow 0$) that satisfies the conditions of continuity of the solute concentration c and its derivative $\partial c/\partial x$. Yet the problem has a self-similar solution of the second kind with a functional representation of the form

$$c(x,t)=A\frac{c(\xi)}{(Dt)^{(1+\alpha)/2}}$$

$$A=\gamma_0\lim_{\delta\Rightarrow 0}c_0\delta^{\alpha},\quad c_0=\int_{-\infty}^{+\infty}c(x,0)\,dx,\quad \xi=\frac{x}{(Dt)^{1/2}} \tag{2.2}$$

where γ_0 is a multiplier that depends on the normalization conditions of $c(\xi)$, and the exponent α is determined by the parameters of the problem. The dependences $\alpha(\varepsilon^+, \varepsilon^-, \beta)(\beta = \lambda/D^{1/2})$ obtained by numerical solution of the non-linear eigenvalue problem were presented in [1, 2]. For $\beta=0$ [$v(t)=0$], this solution is identical with the self-similar solution of the second kind for a modified heat source [6].

Self-similarity of the second kind implies that the function F in (2.2) depends non-analytically on the parameter δ as $\delta\Rightarrow 0$. Therefore, Eq. (1.2) cannot be solved by the ordinary perturbation method with a small parameter ε taking $Q_0G(x, t, -\delta)$ as the initial approximation. Indeed, this process, if it were legitimate, would produce a power series in ε with every term in the form of a self-similar solution of the first kind for $\delta\Rightarrow 0$ (i.e. analytical in δ), and the solution therefore would be analytical in δ. Non-analytical dependence on δ is attributable to the fact that $\partial c/\partial t$ has a non-integrable singularity as $\delta\Rightarrow 0$, and the perturbation-theory corrections therefore contain divergences.

These divergences are similar to the divergences that arise in quantum field theory when regularization is removed [7] (in our problem, δ plays the role of a regularization parameter). The field-theory divergences are eliminated by a renormalization procedure, which ensures that non-analytical dependence on the regularization parameter enters only the renormalization constants of the original system parameters and field amplitudes. In the presence of a dimensional regularization parameter, the renormalization constants are of non-zero dimension, so that the renormalized physical parameters acquire an additional (anomalous) dimension, i.e. they transform in an unusual way under scale transformations [7–9]. The anomalous dimension exponents in field theory are identical with the exponents of partial self-similarity (self-similarity of the second kind) in the intermediate asymptotic (IA) solution method [6]. This phenomenon has been noted in [10]. In field theory, the anomalous dimension exponents are calculated by the RG method, which provides a technique for improving the perturbation-theory results by imposing the condition of renormalization invariance, i.e. a condition ensuring that the computed asymptotic behaviour of a physical quantity is independent of the choice of the normalization conditions [7, 9].

In this paper, the method of calculating the anomalous dimension exponent for the diffusion equation with sorption hysteresis [10] is generalized to the case of diffusion-convective transport. A self-similar solution of the second kind for convective transport has been obtained by the IA method [1, 2] for the case $v=\lambda t^{1/2}$.

3. In accordance with the RG method [1], we rewrite Eq. (1.2) in the integral form

$$c(x,t) = \int dx' G(x-x',t,0)\,c(x',0) -$$
$$- \varepsilon \int dx' \int dt' G(x-x' t,t')\, H\left[-\frac{\partial c(x',t')}{\partial t'}\right]\left[f(t)\frac{\partial}{\partial x'} - \frac{1}{2}\frac{\partial^2}{\partial x'^2}\right] c(x',t')$$

where the first term on the right-hand side is the solution of the unperturbed problem ($\varepsilon = 0$). Substituting the initial condition (1.3), we obtain

$$c^{(0)}(x,t) = \int dx' G(x-x',t,0)\,c(x',0) = Q_0 G(x,t,-\delta) \qquad (3.1)$$

Here we have used the fact that Green's function of the convective-diffusion transport equation (1.4) obviously satisfies the relationship

$$G(x-x_0,t,t_0) = \int dx' G(x-x',t,t')\,G(x'-x_0,t',t_0) \qquad (3.2)$$

which is essentially a Smoluchowski–Kolmogorov–Chapman equation for a Markov process [12].

Iterative solution of the integral equation corresponds to the representation of the solution c as a perturbation-theory series in powers of ε, in which every term diverges as $\delta \Rightarrow 0$. Improvement of the perturbation theory reduces to rearranging this series by renormalization of the coefficient Q_0 in (3.1). To this end, substituting (3.1) into the integral equation, we replace the original parameter Q_0 by the renormalized (phenomenological) parameter $Q = ZQ_0$ and add to the perturbing part a compensating counterterm (CT) of the form $(1-Z)\,Q_0(x,t,-\delta)$. As a result, we obtain

$$\begin{aligned} c(x,t) &= QG(x,t,-\delta) - \varepsilon \int dx' \int dt' G(x-x',t,t') \times \\ &\times H[-\partial c(x',t')/\partial t']\,[f(t)\,\partial/\partial x' - {}^1\!/_2\,\partial^2/\partial x^2] \times \\ &\times c(x',t') + (1-Z)\,Q_0 G(x,t,-\delta) \end{aligned} \qquad (3.3)$$

The renormalization constant Z is defined in such a way that the singular correction to the renormalized Q vanishes as $\delta \Rightarrow 0$ at some "normalization point" $t = \tau$.

We are looking for the asymptotic solution of the problem in the form

$$c(x,t) = q(t,\delta,\varepsilon,Q_0)\,t^{-1/2} F(x,t,\varepsilon),\quad q(t,\delta,\varepsilon,Q_0) = \int dx\, c(x,t) \qquad (3.4)$$

i.e. the non-analytical dependence on δ as $\delta \Rightarrow 0$ only occurs in the function $q(t,\delta,\varepsilon,Q_0)$, which represents the total quantity of the solute at time t (it varies as a result of partial irreversibility of sorption). The function $F(x,t,\varepsilon)$ depends on the self-similar variable $[x - \int f(s)\,ds]^2/t$, but it is no longer a Gaussian exponential.

For $\varepsilon = 0$, we have

$$q = Q_0,\quad F(x,t,0) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2t}\left(x - \int_0^t f(s)\,ds\right)^2\right\} \qquad (3.5)$$

and the constant Q_0 is the total amount of the solute in the fluid, which remains constant. The asymptotic solution corresponding to a thin initial impulse is self-similar.

The renormalization constant $Z = Q/Q_0$ is defined by the condition

$$Q = q(t,\delta,\varepsilon,Q_0)\,|_{t=\tau} \qquad (3.6)$$

By (3.6), the singular dependence on δ is incorporated in the phenomenological parameter Q, which is the amount of impurity at time τ.

When $c(x,t)$ is evaluated by the renormalized perturbation theory, the function $q(t,\delta,\varepsilon,Q_0)$ defined by (3.4) depends in the limit as $t \to \infty$ ($\delta \to 0$) on the parameter $Q = ZQ_0$, the time t, and the choice of the normalization point τ; from dimensional considerations we obtain

$$q(t,\delta,\varepsilon,Q_0) \to q(t,\tau,\varepsilon,Q) = Q\varphi(t/\tau,\varepsilon) \qquad (3.7)$$

Renormalization invariance implies that the physical picture does not change when the timescale τ is replaced by τ_1 and Q is accordingly replaced by Q_1 [the parameter Q_1 is defined by (3.6) for $t = \tau_1$], i.e.

$$Q\varphi(t/\tau,\varepsilon) = Q_1\varphi(t/\tau_1,\varepsilon) \qquad (3.8)$$

By the normalization condition (3.6),

$$\varphi(1, \varepsilon) = 1 \tag{3.9}$$

From (3.8) and (3.9), we obtain the RG functional equation for the function φ:

$$\varphi(u, \varepsilon) = \varphi(\lambda, \varepsilon)\,\varphi(u/\lambda, \varepsilon) \tag{3.10}$$

Differentiating (3.10) with respect to λ and then setting $\lambda = 1$, we obtain the RG differential equation

$$\{-u\partial/\partial u + \alpha_R\}\,\varphi(u, \varepsilon) = 0, \quad \alpha_R = \partial\varphi(u, \varepsilon)/\partial u\,|_{u=1} \tag{3.11}$$

The solution of Eq. (3.11) that satisfies the normalization condition (3.9) has the form

$$\varphi(t/\tau, \varepsilon) = (t/\tau)^{\alpha_R(\varepsilon)} \tag{3.12}$$

and the problem of finding the partial self-similarity exponent reduces to evaluating the function $\varphi(t/\tau, \varepsilon)$ near the normalization point $t = \tau$. By the RG method in its field version [7], we calculate $\varphi(t/\tau, \varepsilon)$ in the lowest approximation of the renormalized perturbation theory, i.e. we take the first iteration for Eq. (3.3)

$$c(x,t) = QG(x,t,-\delta) + \varepsilon Q\int_0^t dt' \int dx' G(x-x', t, t') \times$$
$$\times H[-\partial G(x', t', -\delta)/\partial t']\,\partial G(x', t', -\delta)/\partial t' + \mathrm{CT} \tag{3.13}$$

Using the expression for Green's function (1.3) and definition (3.6), we obtain

$$q(t, \delta, \varepsilon, Q_0) = Q_0 Z\sqrt{\frac{t}{t+\delta}} + \frac{\varepsilon}{\sqrt{8\pi}}\int_0^t \frac{dt'}{t'}\left(\frac{t'}{t+\delta-t'}\right)^{1/2}\int_{w_-(t',\delta)}^{w_+(t',\delta)} dw\, \times$$
$$\times \exp\left\{-\frac{w^2}{2}\frac{t+\delta}{t+\delta-t'}\right\}[w - 1 + 2w\sqrt{t'}\,f(t'+\delta)] +$$
$$+ Q_0(1-Z)\sqrt{t/(t+\delta)}, \quad w_\pm(t, \delta) = \pm\sqrt{tf^2(t-\delta)+1} - tf(t-\delta) \tag{3.14}$$

As $t/\delta \to \infty$, noting that the main contribution to the integral (3.14) for $\delta \Rightarrow 0$ comes from the region of small t', we obtain

$$q(t, \delta, \varepsilon, Q_0) = Q_0 Z\left\{1 + \varepsilon/(8\pi)^{1/2}\int_\delta^t \frac{dt'}{t'}\int_{w_-(t',0)}^{w_+(t',0)} dw\, W(w, t')\right\} + Q_0(1-Z) \tag{3.15}$$
$$W(w, t) = \exp(-w^2/2)[w^2 - 1 + 2w\sqrt{t-\delta}\,f(t)]$$

The parameters of the CT $Q_0(1-Z)$ are chosen so that they satisfy the normalization condition (3.6). As a result, we obtain

$$\varphi(t/\tau, \varepsilon) = 1 + \frac{\varepsilon}{(8\pi)^{1/2}}\int_\tau^t \frac{dt'}{t'}\int_{w_-}^{w_+} dw W(w, t') \tag{3.16}$$

In the self-similar case, when $f(t) = \gamma t^{-1/2}$, the functions $w_\pm$ are independent of t and the integrals are easily evaluated. We obtain

$$\varphi(t/\tau, \varepsilon) = 1 + \varepsilon A \ln(t/\tau) \tag{3.17}$$

$$A = \frac{1}{\sqrt{8\pi}}\int_{w_-}^{w_+} dw \exp\left(-\frac{w^2}{2}\right)[w^2 - 1 + 2w\gamma] =$$
$$= \frac{1}{2\sqrt{2\pi e}}\{e^{\gamma\gamma_-}\gamma_+ + e^{\gamma\gamma_+}\gamma_-\} \cong -\frac{1+\gamma^2}{\sqrt{2\pi e}}, \quad \gamma_\pm = \sqrt{\gamma^2+1} \pm \gamma \tag{3.18}$$

For small ε, we should thus have

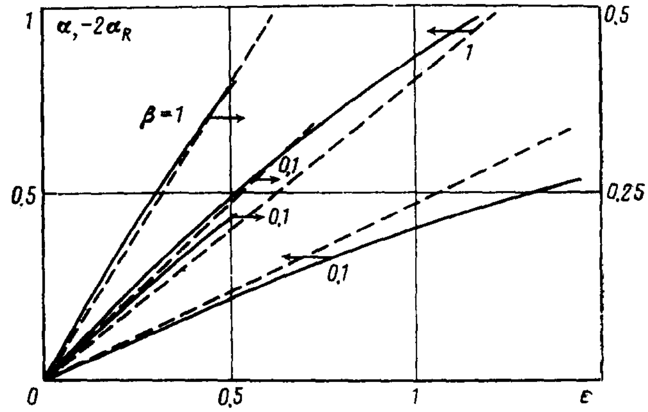

FIG. 1.

$$-2\alpha_R(\varepsilon, \gamma) = \alpha(\varepsilon^+, \varepsilon^-, \beta),\ \gamma = \beta/(2\varepsilon^+)^{1/2},\ \varepsilon = (\varepsilon^+ - \varepsilon^-)/\varepsilon^- \tag{3.19}$$

Figure 1, using the graphs of $\alpha(\varepsilon^+, \varepsilon^-, \beta)$ from [1, 2], compares the values of $2\alpha_R$ (the RG method, the solid curves) and α (the IA method, the dashed lines) as a function of ε for two fixed values of ε^-: $\varepsilon^- = 0.25$ ($m = 0.2$, $\Gamma^- = 0.05$, the left scale) and $\varepsilon^- = 0.4$ ($m = 0.2$, $\Gamma^- = 0.2$, the right scale). Note that for $\varepsilon^- = 0.4$ the values of $-2\alpha_R$ and α are virtually identical. For $\varepsilon^- = 0.25$, for large values of γ and the same values of ε, the calculation of α_R up to the first term of perturbation theory is insufficiently accurate. For $\beta = 0$, the results of calculations from (3.18) are identical with the calculations of α_R in [10].

4. The same method is applicable to the axisymmetric convective diffusion problem with irreversible sorption in a stationary velocity field of the form

$$\mathbf{v}(\mathbf{r}) = \lambda\mathbf{r}/r^2 \tag{4.1}$$

In this case, the concentration equation

$$[\varepsilon^{\pm}\partial/\partial t + \lambda r^{-1}\partial/\partial r - D\Delta^{(2)}]c(r, t) = 0 \tag{4.2}$$

for a radially symmetric initial distribution can be represented like (1.2) in the form

$$\left\{\frac{\partial}{\partial t} + \frac{\beta-1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2}\right\}c(r,t) = \\ = \varepsilon H\left[-\frac{\partial c}{\partial t}\right]\left[\frac{\partial^2}{\partial x^2} - \frac{\beta-1}{r}\frac{\partial}{\partial r}\right]c(r,t),\quad \beta = \frac{\lambda}{D} \tag{4.3}$$

Here we have made the change of variables $r \Rightarrow r(D/\varepsilon^+)^{1/2}$. As in the passage to Eq. (3.3), we rewrite Eq. (4.3) in integral form

$$c(r,t) = \int_0^\infty r'dr'G(r,r',t)c(r',0) + \varepsilon\int_0^t ds\int_0^\infty r'dr'G(r,r',t-s)\times \\ \times H\left[-\frac{\partial}{\partial s}c(r',s)\right]\left[\frac{\partial^2}{\partial r'^2} - \frac{\beta-1}{r'}\frac{\partial}{\partial r'}\right]c(r',s) \tag{4.4}$$

Green's function $G(r, r', t)$ satisfies the equation

$$\{\partial/\partial t + (\beta-1)r^{-1}\partial/\partial r - \partial^2/\partial r^2\}G(r, r', t) = \delta(t)\delta(r-r')r^{-1} \tag{4.5}$$

the solution of which has the form

$$G(r,r',t) = H(t)\left(\frac{r}{r'}\right)^{\beta/2}\int_0^\infty \lambda\,d\lambda\exp(-\lambda^2 t)J_{\beta/2}(\lambda r)J_{\beta/2}(\lambda r') \tag{4.6}$$

where $J_n(x)$ is Bessel's function of the first kind.

Using the well-known Fourier–Bessel integral, we can show that Green's function (4.6) satisfies the Smoluchowski–Kolmogorov–Chapman equation

$$G(r, r_0, t - t_0) = \int_0^\infty r'dr' G(r, r', t - t') G(r', r_0, t' - t_0) \tag{4.7}$$

which is an analogue of relationship (3.2) for the one-dimensional problem.

Taking the initial distribution in the form

$$c(r, 0) = Q_0 G(r, 0, -\delta) = \frac{Q_0}{2\Gamma(1+\beta/2)\delta} \left(\frac{r^2}{4\delta}\right)^{\beta/2} \exp\left(-\frac{r^2}{4\delta}\right) \tag{4.8}$$

and using (4.7), we obtain the solution of the unperturbed problem

$$c^0(r, t) = Q_0 G(r, 0, t + \delta) = \frac{Q_0}{2\Gamma(1+\beta/2)(t+\delta)} \left(\frac{r^2}{4(t+\delta)}\right)^{\beta/2} \exp\left(-\frac{r^2}{4(t+\delta)}\right) \tag{4.9}$$

Equation (4.3) thus takes the form

$$\begin{aligned} c(r, t) = Q_0 G(r, 0, t + \delta) + \varepsilon \int_0^t ds \int_0^\infty r' dr' G(r, r', t - s) \times \\ \times H\left[-\frac{\partial c(r', s)}{\partial s}\right] \left[\frac{\partial^2}{\partial r^2} - \frac{\beta - 1}{r} \frac{\partial}{\partial r}\right] c(r', s) \end{aligned} \tag{4.10}$$

Like the above, we renormalize the parameter Q_0 by making the change $Q_0 \Rightarrow Q = ZQ_0$ and adding a compensating counterterm. Then, in the lowest approximation of the renormalized perturbation theory, we obtain

$$\begin{aligned} c(r, t) = QG(r, r', t + \delta) + \varepsilon \int_\delta^t d \int_0^\infty r'dr' G(r, r', t - s) \times \\ \times H\left[-\frac{\partial G(r', 0, s + \delta)}{\partial s}\right] \frac{\partial G(r', 0, s + \delta)}{\partial s} + \mathrm{CT} = QG(r, 0, t + \delta) + \\ + \varepsilon Q \int_\delta^t \frac{ds}{s^2} \int_0^{\sqrt{(1+\beta/2)s}} r' dr' G(r, r', t + \delta - s) \frac{1}{2\Gamma(1+\beta/2)} \left(\frac{r'^2}{4s}\right)^{\beta/2} \exp\left(-\frac{r^2}{4\delta}\right) \times \\ \times \left[1 + \frac{\beta}{2} - \frac{r'^2}{4s}\right] + \mathrm{CT} = Q\Big\{G(r, 0, t + \delta) - \varepsilon Q \int_\delta^t \frac{ds}{s} \int_0^{\sqrt{1+\beta/2}} \xi \, d\xi \times \\ \times G(r, \sqrt{s}\,\xi, t + \delta - s) \zeta(\xi) + (Z^{-1} - 1) G(r, 0, t + \delta)\Big\}, \\ \zeta(\xi) = \frac{1}{2\Gamma(1+\beta/2)} \left(\frac{\xi^2}{4}\right)^{\beta/2} \exp\left(-\frac{\xi^2}{4}\right)\left(1 + \frac{\beta}{2} + \frac{\xi^2}{4}\right) \end{aligned} \tag{4.11}$$

As $\delta \Rightarrow 0$, the main contribution to the integral over s is from the region near $s = 0$, where the integrand is singular. Retaining in (4.11) only the contribution from the singular part and choosing the renormalization constant Z from the normalization condition (3.6), we obtain for the total quantity of the solute

$$\begin{aligned} q(t, \tau, \varepsilon) = Q\{1 - \varepsilon A \ln(t/\tau)\} \\ A = \int_0^{2\sqrt{1+\beta/2}} \xi \, d\xi \zeta(\xi) = \frac{[(1+\beta/2)/e]^{1+\beta/2}}{\Gamma(1+\beta/2)} \end{aligned} \tag{4.12}$$

Thus, for the partial self-similarity exponent we obtain in our problem

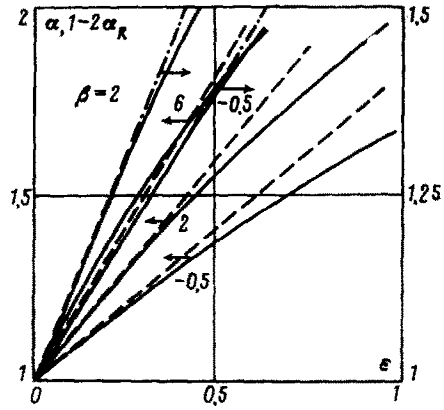

FIG. 2.

$$\alpha_R = -\varepsilon A \tag{4.13}$$

The comparison of the self-similarity exponents $1-2\alpha_R$ (calculated by the RG method) and $\alpha(\varepsilon^+, \varepsilon^-, \gamma)$ (calculated by the IA method [1]) is presented in Fig. 2. The results are close to those for the convective diffusion equation in the one-dimensional case. Here the left scale corresponds to $m = 0.2$, $\Gamma^- = 0.05$, and the right scale to $m = 0.2$, $\Gamma^+ = 0.2$.

The method of Sec. 4 can also be used to solve the diffusion equation with irreversible sorption (without convection) in the d-dimensional case. As a result, for the partial self-similarity exponent we obtain the expression

$$\alpha_R = -\varepsilon\,(d/(2e))^{d/2}/\Gamma\,(d/2) \tag{4.14}$$

which in the one-dimensional case $d = 1$ reproduces the previous result of [10].

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